A SURVAY OF COURNOT OLIGOPOLY DYNAMICS

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Abstract

In this paper a Cournot-like models are constructed with an iso-elastic demand function for n competitors with respect to the homogenity and some adjustment aspects. The Cournot equilibriums are constructed for general constant unit costs, and also two cycle is detected for adjust Cournot oligopoly model. Moreover, the stability of Cournot points is discussed in detail. That means, it is pointed out the influence of heterogenity on the stabilization of the stability of the Cournot point under increasing number of firms.

Keywords

Cournot Equilibrium, Dynamical System, Sink, Source, Saddle

I. Introduction

There are two opposing market forms in economics: competition and monopoly. In the case of competition, firms are numerous and hence small in comparison to the size of the total market, and they consider the market price to be exogenously determined. In a monopoly, only one firm supplies the market and supply influences the market price appreciably.

An *oligopoly* is a market form in which a market has a dominant influence on a small number of sellers (oligopolists). Since there are few sellers, each oligopolist is likely to be aware of the actions of the others. Each seller has an influence on, and is influenced by, the decisions of the other oligopolists. Hence, the planning of each oligopolist needs to take into account the responses of the other competitors. In an oligopoly, there are at least two firms controlling the market. If there are two sellers, it is called a *duopoly*; while if there are three competitors, it is known as a *triopoly*. The first treatment of oligopoly was proposed by A. Cournot, in 1838 [14], for a duopoly. Significant additions to the theory were made exactly one hundred years later by H. von Stackelberg [33].

A question in microeconomics is whether an increase in the number of competitors in a market defines a path to perfect competition. It was stated by [34] (see also [26] page 237) that the oligopoly model produced under constant marginal costs with a linear demand function is neutrally stable for three competitors and unstable for more than three competitors. The argument for this fact can be found in [30]. As discussed in [30], linear demand functions are very easy to use, but they do not avoid negative supplies and prices, so it is possible to use them only for the study of local behavior. This problem can be solved by using nonlinear demand functions such as piecewise linear functions or other more complex functions, one of which was suggested by [27] for a dupoly and later by [28] for a triopoly using iso-elastic demand functions. These types of demand function were later studied by [1] and [3] for a nonlinear (iso-elastic) demand function and constant marginal costs and it was concluded that this Cournot model for n competitors is neutrally stable if $n = 4$ and is unstable if the number of competitors is greater than five (see also [30] and [20]).

Furthermore, in [5] an evolutionary model of oligopoly competition where agents can select between different behavioral rules to make decisions on productions. In [7] a nonlinear discrete-time dynamic

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model proposed by [16] as a market share attraction model with two firms that decide marketing efforts over time according to best reply strategies with naive expectations. In [6] the authors researched mathematical properties and dynamics of a duopoly with price competition and horizontal product differentiation by introducing quadratic production costs (decreasing returns to scale), thus extending the model with linear costs (constant returns to scale) of [17]. The main aim of [19] is a study of local and global dynamics in a nonlinear duopoly with quantity-setting firms and non-cooperative advertising in vestments that affect the degree of (horizon-tally) differentiated products. The paper [18] extends the classical repeated duopoly model with quantity-setting firms of [9] by assuming that production of goods is subject to some gestation lags but exchanges take place continuously in the market.

Opposing to the previous results reached on homogeneous Cournot models the following are reached on heterogeneous one, such results are not so numerous. The [11] paper studies oligopolies of generic size consisting of heterogeneous firms, which adopt best response adjustment mechanisms with either perfect foresight (rational firms) or static expectations (naive firms). [13] develop a heterogeneous agents model of as set price and inventory with a market maker who considers the excess demand of two groups of agents that employ the same trading rule (i.e. fundamentalists) with different beliefs on the fundamental value. The work [10] studies oligopoly models in which firms adopt decision mechanisms based on best response techniques with different rationality degrees. Heterogeneous Cournot oligopolies of variable sizes and compositions, in which the firms have different degrees of rationality, being either rational firms with perfect foresight or naive best response firms with static expectations are explored in [12]. The authors in [32] consider a Schelling-type segregation model with two groups of a gents that differ in some aspects, such as religion, political affiliation or color of skin.

The main aim of this paper is to consider Cournot oligopoly models, form classical homogeneous to heterogeneous, and to construct the Cournot points and discuss their stability while the number of players is increasing. The terminology of dynamical systems is used, that is the Cournot point is identified as a fixed one.

A discrete dynamical system is an ordered pair (X, f) where X is standardly taken to be a compact metric space and f is a continuous map from X to, but not necessarily onto, X. So, X is *invariant* under f, that is $f(X) \subset X$. A point $x \in X$ is *fixed* if $f(x) = x$ and a set of all fixed points of the map f is denoted by $\mathrm{Fix}(f)$. For $t \in \mathbb{N}$, the *t-th iterate* of f is the t-fold composition $f^t = f \circ \ldots \circ f$, where f^0 is the identity map.

The paper is organized as follows. In the second section the Cournot iso-elastic model with n competitors is derived, and the model is constructed as a discrete dynamical system ([illegible]$_n$, F_n). In the third section it is shown that the Cournot point is a sink for $n = 2, 3$ and is a saddle for $n > 4$. It is known that for $n = 4$ the Cournot point is neutrally stable (see e.g. [1] or [30]). In the fourth section an adjust Cournot model is derived, its Cournot point is detected and also two cycle is shown, some open problems are stated. Finally, in the fifth section a heterogeneous Cournot model with respect to the different kinds of players: local monopolistic approximation (LMA), Nash and rational players are considered. It is pointed out that this type of Cournot model preserves the stability of the Cournot point while increasing the number of players.

II. The Cournot iso-elastic model for N competitors

The following construction is inspired by the work of T. Puu [27] who constructed the model for two competitors (later in [28] for three players). This model could be extended for n firms.

Assuming that the level of demand is reciprocal to price p, this represents an "iso-elastic" demand function reflecting a case where consumers always spend a constant sum on the commodity, regardless of price. Inverting the demand function gives

$$p = \frac{1}{x_1 + x_2 + \cdots + x_n}, \tag{1}$$

where the total quantity in the denominator is the sum of the supplies and x_i are competitors, for $i \in \{1, 2, \ldots, n\}$.
The revenues of these firms equal price times quantity

$$px_i = \frac{x_i}{x_1 + x_2 + \cdots + x_n}. \tag{2}$$

Assuming that the firms operate under constant unit costs c_i, so $c_i > 0$ for any i. Their total costs are $c_i x_i$. So, the profits become

$$\Pi_i(x_1, x_2, \ldots, x_n) = \frac{x_i}{x_1 + x_2 + \cdots + x_n} - c_i x_i. \tag{3}$$

In order to maximize profits for the firm put a partial derivative of (3) with respect to x_i equal to zero

$$\frac{\partial \Pi_i}{\partial x_i} = 0, \tag{4}$$

and these *reaction functions* are derived

$$\begin{aligned}
x_1(x_2, x_3, \ldots, x_n) &= \sqrt{\frac{x_2 + x_3 + \ldots + x_n}{c_1}} - (x_2 + x_3 + \ldots + x_n), \\
x_2(x_1, x_3, \ldots, x_n) &= \sqrt{\frac{x_1 + x_3 + \ldots + x_n}{c_2}} - (x_1 + x_3 + \ldots + x_n), \\
&\vdots \\
x_n(x_2, x_3, \ldots, x_{n-1}) &= \sqrt{\frac{x_2 + x_3 + \ldots + x_{n-1}}{c_n}} - (x_2 + x_3 + \ldots + x_{n-1}).
\end{aligned} \tag{5}$$

Introducing the adjustment process explicitly, from (5) the following are obtained

$$\begin{aligned}
x_1^t &= \sqrt{\frac{x_2^{t-1} + x_3^{t-1} + \ldots + x_n^{t-1}}{c_1}} - (x_2^{t-1} + x_3^{t-1} + \ldots + x_n^{t-1}), \\
x_2^t &= \sqrt{\frac{x_1^{t-1} + x_3^{t-1} + \ldots + x_n^{t-1}}{c_2}} - (x_1^{t-1} + x_3^{t-1} + \ldots + x_n^{t-1}), \\
&\vdots \\
x_n^t &= \sqrt{\frac{x_2^{t-1} + x_3^{t-1} + \ldots + x_{n-1}^{t-1}}{c_n}} - (x_2^{t-1} + x_3^{t-1} + \ldots + x_{n-1}^{t-1}).
\end{aligned} \tag{6}$$

Thus, a dynamical system is deduced

$$(\text{[illegible]}_n, F_n) \tag{7}$$

defined as follows for $n \geq 2$.
Denote

$$\text{[illegible]} = (x_1, x_2, x_3, \ldots, x_n)$$

and define operator $({}^i\bullet)$

$$({}^i\,\text{[illegible]}) = \sum_{j \in N_i} x_j$$

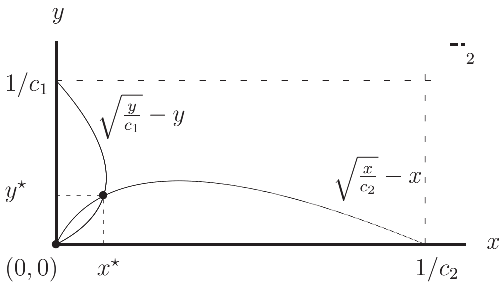

Figure 1: Cournot equilibrium in the second dimension

where $N = \{1, 2, 3, \ldots, n\}$ and $N_i = N \setminus \{i\}$.
Firstly, set

$$F_n : \mathbb{R}^n \to \mathbb{R}^n$$

defined by

$$\mathbf{x} \mapsto \left(\sqrt{\frac{(^1\mathbf{x})}{c_1}} - (^1\mathbf{x}), \sqrt{\frac{(^2\mathbf{x})}{c_2}} - (^2\mathbf{x}), \ldots, \sqrt{\frac{(^n\mathbf{x})}{c_n}} - (^n\mathbf{x}) \right).$$

Secondly, the invariant set Ω_n is found. It is clear that the domain of F_n is

$$D_n = \left\{ (^i\mathbf{x}) \geq 0, \ \text{for any} \ i \right\}.$$

Such a model is being considered so that the process can be repeated, thus focussing on a subset of D_n for which $F_n^t(\mathbf{x}) \in D_n$ for any $t \geq 0$. Such points in D_n are *admissible* (see [4]). Unfortunately, not all admissible points are meaningful. Since economic interpretation of a point with negative value is not acceptable the attention has to be restricted to *feasible* points. The first iteration needs to be taken into account. To do this, this system of equations has to be solved

$$(^i\mathbf{x}) = \frac{1}{c_i}, \tag{8}$$

Solving system (8) gives the solution

$$x_i = \frac{1}{n-1} \left(^i(1/\ \ ) \right) + (2-n)c_i, \tag{9}$$

where $1/\ \ = (1/c_1, 1/c_2, \ldots, 1/c_n)$. Denoting x_i by a_i, a set is obtained, such that all their points are feasible for the first iteration. Repeating this process for all iterations gives

$$\Omega_n \subset [0, a_1] \times [0, a_2] \times \ldots \times [0, a_n] \subset D_n \tag{10}$$

which is invariant under F_n, see Figure 1 for the second-dimensional case. Unfortunately, it is not easy to ascertain the set Ω_n explicitly.

Notice that

$$F_n(a_1, a_2, \ldots, a_n) = (0, 0, \ldots, 0),$$

and that the spaces Ω_3, Ω_4 are not well constructed e.g. in [21] and [22]. The space $\Omega_2 = [0, 1/c_2] \times [0, 1/c_1]$ is implicitly given in [35], and this can be compared with (9) and (10). It is also easy to see that a fixed point $\mathbf{x}^\star$ of F_n is the *Cournot equilibrium* (Nash equilibrium of the game) (see e.g. [4]).

III. Stability of the Cournot equilibrium of the Cournot iso-elastic model

A fixed point p for $f : \mathbb{R}^n \to \mathbb{R}^n$ is called *hyperbolic* if $Df(p)$ has no eigenvalues on the unite circle, where $Df(p)$ is the Jacobian matrix of f at the point p. Such a hyperbolic point p is

1. a *sink* fixed point if all eigenvalues of $Df(p)$ are less than one in absolute value,
2. a *source* fixed point if all eigenvalues of $Df(p)$ are greater than one in absolute value,
3. a *saddle* fixed point otherwise, i.e., if some eigenvalues of $Df(p)$ are less and some larger than one in absolute value.

Proposition 1 ([15]) *Supposing that $f : \mathbb{R}^n \to \mathbb{R}^n$ has a sink fixed point p. Then there is an open set containing p in which all points tend to p under forward iteration of f.*

The largest such open set in $\mathbb{R}^n$ is called the *stable set* of p and is denoted by $W^s(p)$.

Proposition 2 ([15]) *Supposing that $f : \mathbb{R}^n \to \mathbb{R}^n$ has a source fixed point p. Then there is an open set containing p in which all points tend to p under backward iteration of f.*

The largest such open set in $\mathbb{R}^n$ is called the *unstable set* of p and is denoted by $W^u(p)$.

The stability of the Cournot point has been discussed for two or three players. The case for two competitors was discussed in [35]. For $n = 2$

$$\text{Fix}(F_2) = \left\{(0,0), \left(\frac{c_2}{(c_1+c_2)^2}, \frac{c_1}{(c_1+c_2)^2}\right)\right\}.$$

In [4] the situation is for $n = 3$

$$\text{Fix}(F_3) = \left\{(0,0,0), \left(\frac{2(-c_1+c_2+c_3)}{(c_1+c_2+c_3)^2}, \frac{2(c_1-c_2+c_3)}{(c_1+c_2+c_3)^2}, \frac{2(c_1+c_2-c_3)}{(c_1+c_2+c_3)^2}\right)\right\}.$$

Note that the first part of Theorem 2 in [22] is not correct. The mistake is in the section "Proof of Theorem 2". Taking $c_3 = c_1 + c_2$ it is concluded "does not provide any fixed point different from the origin". Yet, the second fixed point is $(c_2/(c_1+c_2)^2, c_1/(c_1+c_2)^2, 0)$.

Theorem 3 ([24]) *Let $n \geq 2$ and $(\ {}_n, F_n)$ be a dynamical system (13). Then*

$$\text{Fix}(F_n) = \{\ {}_0,\ {}_1\}$$

where

$$
\begin{aligned}
{}_0 &= (0,0,\ldots,0),\\
{}_1 &= \left(\frac{(n-1)({}_1\ )}{{}^2}, \frac{(n-1)({}_2\ )}{{}^2}, \ldots, \frac{(n-1)({}_n\ )}{{}^2}\right),\\
&= \sum_{j\in\{1,2,\ldots,n\}} c_j,\\
({}_i\ ) &= \sum_{j\in N_i} c_j + (2-n)c_i.
\end{aligned}
$$

From now on it is assumed that the oligopolists' products have identical marginal costs, that is $c_i = c$ for any i. Then using Theorem 3 the following is deduced

$$\mathrm{Fix}(F_n) = \left\{(0,0,\ldots,0), \left(\frac{(n-1)}{cn^2}, \frac{(n-1)}{cn^2}, \ldots, \frac{(n-1)}{cn^2}\right)\right\}.$$

Theorem 4 ([24]) *If $c_i = c$ for any i, then for a dynamical system $(\;_n, F_n)$ (13):*

(i) $\dim W^u(\;_0) = n$ *and* $\dim W^s(\;_0) = 0$ *for any* $n \geq 2$,

(ii) $\dim W^u(\;_1) = 0$ *and* $\dim W^s(\;_1) = n$ *for* $n = 2, 3$,

(iii) for $n = 4$ *the point* $\;_1$ *is not hyperbolic, moreover* $\dim W^s(\;_1) \geq 3$,

(iv) $\dim W^u(\;_1) = 1$ *and* $\dim W^s(\;_1) = n - 1$ *for any* $n > 4$.

Corollary 5 ([24]) *The fixed point* $\;_0$ *of* F_n *is a source for any* $n \geq 2$ *and* $c_i = c$.

Corollary 6 ([24]) *The Cournot point* $\;_1$ *of* F_n *is a sink for any* $n = 2, 3$ *and* $c_i = c$.

Corollary 7 ([24]) *The Cournot point* $\;_1$ *of* F_n *is a saddle for any* $n > 4$ *and* $c_i = c$.

IV. Adjustment of the Cournot iso-elastic model

The system (6) indicates that frms in each period, only rely on their knowledge about their own marginal cost and the competitor's behavior in the previous period. Assuming that the competitors adjust their previous decisions by different adjustment speeds given by λ, and μ (both parameters belong to the unit interval), the previous system taken in the second dimension is extended to a system of nonlinear coupled maps given by

$$\begin{aligned} x_1^t &= x_1^{t-1} + \lambda\left(\sqrt{\frac{x_2^{t-1}}{c_1}} - (x_1^{t-1} + x_2^{t-1})\right), \\ x_2^t &= x_2^{t-1} + \mu\left(\sqrt{\frac{x_1^{t-1}}{c_2}} - (x_1^{t-1} + x_2^{t-1})\right). \end{aligned} \tag{11}$$

T. Puu in [29] used system (11) to establish the stability condition of the Cournot point and visualized the emergence of different types of complex dynamics such as fixed points, cycles and chaos. Furthermore, he extended the framework of the analysis to three competing oligopolists and proceeded with addressing similar dynamic features of the model.

Numerical simulations indicate that, for significantly large portions of the parameter space, the system given by (11), leads to negative quantities (in other words trajectories leave the positive part of the real plane). Obviously, negative quantities do not make sense (from an economic viewpoint) in the present context and therefore system (11) should always be stated as:

$$\begin{aligned} x_1^t &= \max\left\{0, x_1^{t-1} + \lambda\left(\sqrt{\frac{x_2^{t-1}}{c_1}} - (x_1^{t-1} + x_2^{t-1})\right)\right\}, \\ x_2^t &= \max\left\{0, x_2^{t-1} + \mu\left(\sqrt{\frac{x_1^{t-1}}{c_2}} - (x_1^{t-1} + x_2^{t-1})\right)\right\}. \end{aligned} \tag{12}$$

Thus, a dynamical system is deduced

$$(\;_2, G_2) \tag{13}$$

on a suitable state space, and given by (12).
It is intuitively clear that the asymptotic behavior of the system (12) is significantly different from the asymptotic behavior of the system (11). From an economic point of view it is interesting to determine how the parameters of the model (12) influence the amplitude and the periodicity of different cycles in the competition process.

Theorem 8 *Let* $(\;_2, G_2)$ *be a dynamical system given by (12). Then*

$$\mathrm{Fix}(G_2) = \{\;_0, \;_1\}$$

where

$$\;_0 = (0,0),$$

and

$$\;_1 = \left(\frac{c_2}{(c_1+c_2)^2}, \frac{c_1}{(c_1+c_2)^2}\right).$$

The proof of Theorem 8 can be done by direct computation, solving the equation $G_2(x_1, x_2) = (x_1, x_2)$. Both above stated fixed points $\;_0$ and $\;_1$ are stable that can be proved directly, see e.g. [29].

Theorem 9 *Let* $(\;_2, G_2)$ *be a dynamical system given by (12). Then there are only two points*

$$\;_0 = \left(0, \frac{c_1}{(c_1+c_2)^2}\right)$$

and

$$\;_1 = \left(\frac{c_2}{(c_1+c_2)^2}, 0\right)$$

such that

$$G_2^2(\;_0) = G_2(\;_1) = \;_0$$

for $\lambda = \mu = 1$.

That means, the points $\;_0$ and $\;_1$ form a two one cycle of the system $(\;_2, G_2)$. The remaining question is whether this cycle is stable or unstable.

Remark 10 *It is worthy to note that the system* $(\;_2, G_2)$ *equals to* $(\;_2, F_2)$ *for* $\lambda = \mu = 1$.

Let $f^n(p) = p$, that is the point p is periodic with prime period n.

1. p is a *sink* or *attracting periodic point* if all eigenvalues of $Df^n(p)$ are less than one in absolute value,

2. p is a *source* or *repelling periodic point* if all eigenvalues of $Df^n(p)$ are greater than one in absolute value,

3. p is a *saddle* point otherwise, i.e., if some eigenvalues of $Df^n(p)$ are less and some larger than one in absolute value.

So, to get the information about stability of $\cdot_0$ and $\cdot_1$ one have to compute Jacobian of the second iteration of G_2

$$JG_2^2(x_1,x_2) = \begin{pmatrix} \frac{\frac{1}{2\,c_2\,\sqrt{\frac{x_1}{c_2}}}-1}{2\,\sqrt{c_1}\,\sqrt{\sqrt{\frac{x_1}{c_2}}-x_1}} - \frac{1}{2\,\sqrt{c_2}\,\sqrt{x_1}} + 1 & 0 \\ 0 & \frac{\frac{1}{2\,c_1\,\sqrt{\frac{x_2}{c_1}}}-1}{2\,\sqrt{c_2}\,\sqrt{\sqrt{\frac{x_2}{c_1}}-x_2}} - \frac{1}{2\,\sqrt{c_1}\,\sqrt{x_2}} + 1 \end{pmatrix}$$

in $\cdot_0$ and $\cdot_1$. The difficulty is that $JG_2^2(\cdot_0)$ and $JG_2^2(\cdot_1)$ contain elements where problem of division by zero occure. Hence, the above stated test is not applicable and an alternative technique have to be utilized, and an open problem can be stated:

Open problem 11 *Are the foregoing periodic points $\cdot_0$ and $\cdot_1$ stable or unstable?*

An analytical way of the proof is not known to the author at the moment, but computational simulations are showing the stability of both points. Hence the natural question arose:

Open problem 12 *Is the set of periodic points of the system $(\cdot_2, G_2)$ dense in the state space?*

The solution of the above stated problem is essential for the complex understanding of the behavior of all points. Nevertheless, computational simulation are showing that the basin of attraction is containing only four points, fixed ones $_0$, $_1$ and the two cycle $\cdot_0$, $\cdot_1$, see Figure 2.

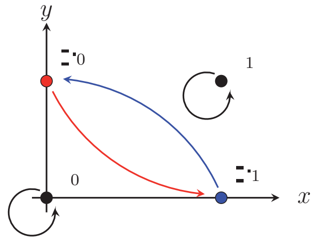

Figure 2: Fixed points and two cycles of $(\cdot_2, G_2)$ for $\lambda = \mu = 1$ and $c_1 = 0.8$, $c_2 = 1.2$.

V. Heterogeneous Cournot oligopoly models

In the previous Sections II., III., and IV. homogeneous oligopolies were introduced and the stability of the Cournot point was discussed. In that cases, the number of firms has always a destabilizing role, the is stability of Cournot point depends on the number of players on the market. The only example of an oligopoly in which the steady state does not lose local stability as the number of firms increases has been proposed in [31] for a homogeneous oligopoly, in which however capacity limits are incorporated in cost functions.

To avoid the problem of destabilization role of the number of players the heterogeneous oligopoly model has been introduced in [10, 11] (see also [12]). In the following the model will be derived. Moreover, each firm is supposed to know the oligopoly composition and the oligopoly size N, which are both constant in time. In what follows we consider three different kinds of players: local monopolistic approximation (LMA), Nash and rational players. LMA players adopts the so-called local monopolistic approximation, which is a boundedly rational mechanism [8]. Nash players have complete information about the price function and the cost functions and make the assumption that all the other players have perfect foresight and use a best response mechanism. Moreover, they have enough computational capabilities to compute the Nash equilibrium (15). Finally, rational players know the demand function, the cost functions of each player and are able to compute the optimal output level which maximizes their profits with respect to the expected strategies of the other players.

So, let us construct a market consisting of n firms, $i = 1,\ 2,\ \ldots n$, producing quantities q^i of homegeneous goods having linear cost functions $C(q^i) = cq^i$, here c denotes identical constant marginal costs of each player. Assume, that the price of the goods are depending on total output Q through the iso-elastic inverse demand function $p(Q) = 1/Q$. Hence the profit of the firm is

$$\pi_i(q^i, Q^{-1}) = \frac{q^i}{q^i + Q^{-1}} - cq^i \tag{14}$$

where Q^{-i} indicates the aggregate output level of all the oligopolists but the i-th one.
This game has only one Nash equilibrium (see [2, 25]) $(q^*, ..., q^*) \in \mathbb{R}^n$ with

$$q^* = \frac{n-1}{n} \tag{15}$$

while the equilibrium aggregated quantity Q^*, profit π^* and price p^* are

$$Q^* = \frac{n-1}{n},\ \ \pi^* = \frac{1}{n^2},\ \ p^* = \frac{n}{n-1}.$$

Now consider a family of oligopolies consisting of heterogeneous combinations of LMA and Nash firms, referred as LN.

Theorem 13 ([10]) *Concerning LN oligopolies, for any size n there exists a heterogeneous composition consisting of a suitably large number of Nash firms for which the steady state is locally asymptotically stable. In particular*

1. *if $n = 2,\ 3,\ 4,\ 5$ then q^* is locally stable for any heterogeneous oligopoly composition;*

2. *if $n = 6,\ 7$ then q^* is locally stable provided that we have at most 4 LMA firms;*

3. *if $n \geq 8$ then q^* is locally stable provided that we have no more than 3 LMA firms.*

The next family of heterogeneous oligopolies is composed by LMA and rational firms, denoted as LR. In this case we have the following.

Theorem 14 ([10]) *Concerning LR oligopolies, for any size n there exists a heterogeneous composition consisting of a suitably large number of rational firms for which the steady state is locally asymptotically stable. In particular*

1. *if $n \leq 7$ then (q^*, q^*) is locally stable for any heterogeneous oligopoly composition;*

2. *if $n \geq 8$ then (q^*, q^*) is locally stable provided that we have at least $\lceil (n3)/4 \rceil$ rational firms.*

VI. Conclusion

This paper concentrates on the Cournot like models and the stabilization role of the number of firms on the stability of the Cournot point. An effort is focused on homogeneous and heterogeneous cases.

In the case of homogeneous models the Cournot point was constructed for ([illegible]$_n$, F_n) and for general unit costs in Theorem 3 and its stability discussed under the additional assumption that the unit costs are identical for all firms in Theorem 4. The Cournot point is a sink for two and three competitors according to Corollary 6 and is a saddle for more than four competitors by Corollary 7, under the assumption of constant marginal costs. It is known that for $n = 4$ the Cournot point is neutrally stable (see e.g. [1] or [30]). Unfortunately, it is not so easy to discuss stability for n competitors in the general case (c_i are different for each i); the situation is more complex and needs in-depth study for each case. The general situation for a triopoly was studied in [4]. It is possible to construct a model "à la Cournot" with the assumption that the firms compete with their closest competitors in either direction. Such systems are equal to ([illegible]$_n$, F_n) for $n = 2, 3$ see.e.g. [23]. The system for four competitors was introduced in [21] where (x_1, x_2, x_3, x_4) is mapped to (y_1, y_2, y_3, y_4) and

$$y_1 = \sqrt{\frac{x_2 + x_4}{c_1}} - (x_2 + x_4), \quad y_2 = \sqrt{\frac{x_1 + x_3}{c_2}} - (x_1 + x_3),$$

$$y_3 = \sqrt{\frac{x_2 + x_4}{c_3}} - (x_2 + x_4), \quad y_4 = \sqrt{\frac{x_1 + x_3}{c_4}} - (x_1 + x_3).$$

It is also possible to extend this idea to dimension n and get Cournot equilibria under the assumption $c_i = c$ for any i. In this case

$$\mathrm{Fix}(F_n) = \left\{ (0, 0, \ldots, 0), \left(\frac{2}{9c}, \frac{2}{9c}, \ldots, \frac{2}{9c} \right) \right\}.$$

In the case of heterogeneous Cournot game the set of results are not so numerous. Namely, the results of [10] are listed in Theorems 13 and 14 that is the only case where the change of firm number does not play a destabilization role on the stability of the Cournot point.

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